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Applying Secant Lines and Tangent Lines for the Optimal Placement of Range Lights

Range lights are pairs of lighthouses on bays, rivers, and other waterways that guide boats safely along a linear path, called the *range line*. Because lighthouses are expensive to build, operate, and maintain, it is reasonable to minimize the number of lighthouses in use by ensuring that the range lines are as long as possible. The portion of a river that is deep enough for boat traffic is referred to as the river's channel. The channel is dredged often to ensure an appropriate depth for safe passage of boats. To keep a boat in a channel bounded by the curves $y = f(x)$ and $y = f(x) + h$, the optimally long range lines are the secant or tangent lines to the two curves. For example, a boat traveling from left to right on the river shown in **figure 1** should stay in the shaded channel.

This department is designed to provide in reproducible formats activities appropriate for students in grades 7–12. The material may be reproduced by classroom teachers for use in their own classes. Readers who have developed successful classroom activities are encouraged to submit manuscripts, in a format similar to "Activities" already published. Of particular interest are activities focusing on NCTM's curriculum standards, its expanded concept of basic skills, problem solving and applications, and the uses of calculators and computers. Send submissions to "Activities" by accessing **mt.msubmit.net**.

Another source of activities can be found in NCTM's *Using Activities from the "Mathematics Teacher" to Support Principles and Standards*, edited by Kimberley Girard and Margaret Aukshun (order number 12746; $35.95), which also includes a grid to help teachers choose the activities that best meet the needs of their students.

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Range lines are comparable to the lines made by connecting two consecutive channel markers that indicate the channel's limits or boundaries. For boats traveling upstream or from open waters, red buoys or red triangular signs mark the channel's left (port) boundary. Green buoys or green square signs mark the channel's right (starboard) boundary. For visibility at night, the markers often have flashing lights of the same color. To ensure that the water is deep enough for safe travel, the boat must remain in the channel between the sets of markers. Both range lights and channel markers use line segments to approximate curves. Range lights use segments to provide a path between the channel's boundaries, while channel markers use segments to provide approximations of the boundaries themselves.

Aerial views of the paths taken by runners on a marathon course or paths taken by race cars on a sinuous race course provide other examples of line segments that create paths between two boundary curves. The boundary curves define the courses and the paths between the curves that are acceptable routes. However, runners and race cars follow straight lines between the two boundary curves because straight lines minimize the distance traveled.

An application such as the range light activity can connect with students on many levels, from the practical to the theoretical. Practical applications often reinforce theoretical ideas in students' minds, emphasizing the utility of mathematics. Being able to apply basic mathematical concepts to real-life situations changes the way mathematics is taught in schools. Making problems interesting and meaningful to students is an everyday challenge. A compendium of diverse applications improves not only the quality of instruction but also the quality of learning.

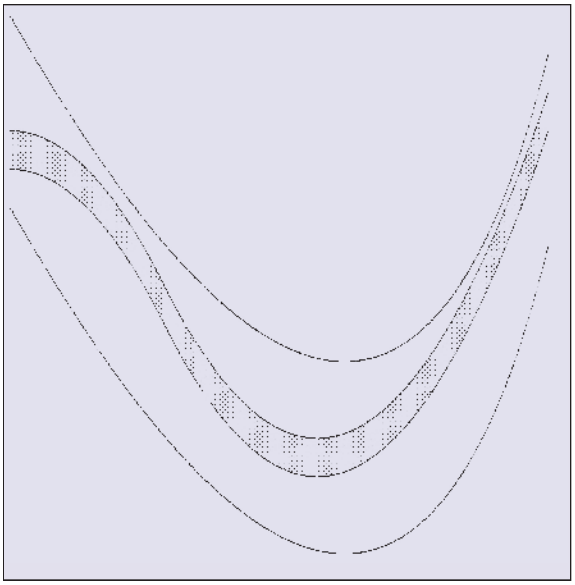
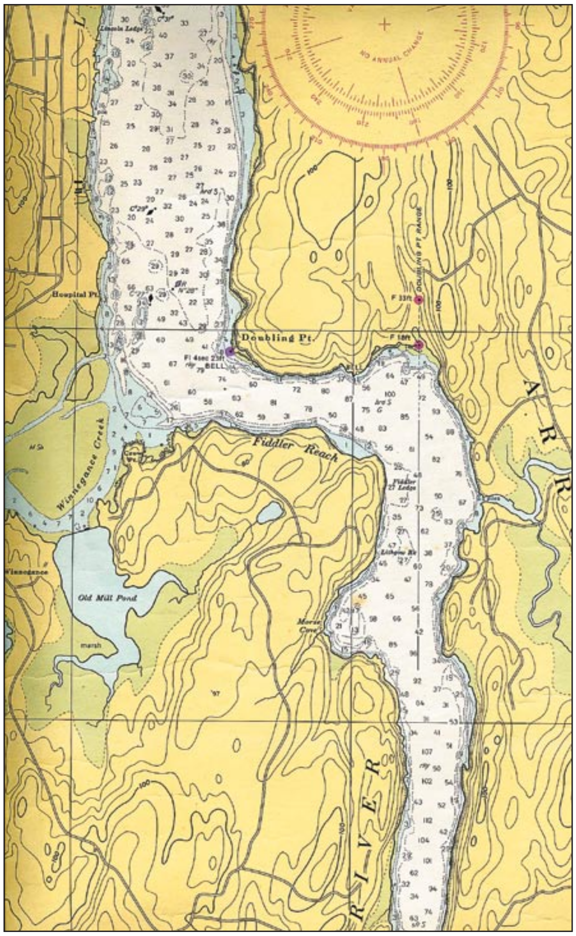


Fig. 1 To ensure safe passage, a boat stays within the shaded channel as it navigates a river. This is river 2's channel, shown on **sheet 2**.



The Problem Solving Standard set forth in *Principles and Standards for School Mathematics* (NCTM 2000) emphasizes the importance of linking classroom activities with real-life problems. The Standard states: "By learning problem solving in mathematics, students should acquire ways of thinking, habits of persistence and curiosity, and confidence in unfamiliar situations that will serve them well outside the mathematics classroom" (p. 52). The range light activity requires students to approximate curves by line segments. In doing so, students use knowledge of secant lines, tangent lines, and first-order derivatives to solve the real-life problem of navigating a boat safely through a channel. This activity highlights the idea that, as the Standard notes, "to meet new challenges in work, school, and life, students will have to adapt and extend whatever mathematics they know" (p. 334).

The range light project requires students to be familiar with the definition of a secant line, the relationship between the derivative and the slope of the tangent line, and derivatives of polynomials. The project reinforces concepts of slope and demonstrates the importance of concavity. As an extension, the process of finding optimal range lines allows teachers to introduce the mean value theorem. In addition to the **activity sheets (1–3)**, students should have a ruler and a calculator. Extra copies of the figures from **sheet 3** are useful, especially if students work in ink. A total time of approximately 1 hour and 15 minutes is appropriate to complete the activity sheets.

This project has been used in a first-year calculus course. A ten-minute explanation of the problem served as an introduction. Including the introduction, students spent 30 minutes on each of two consecutive days on the project, working on **sheet 1** on the first day and **sheet 2** on the second day. Students completed the sheets as part of their homework. Using elementary concepts of slope, such as the rise-over-run definition and the point-slope form of a line, presented more problems for students than taking derivatives. For optimal placement of the first range line, students had to recognize that the range line was tangent to a curve and that its slope had to be calculated in two ways: (1) by using the rise-over-run formula and (2) by using the derivative of a function evaluated at a point. To determine subsequent range lines, students had to find the equation of tangent lines and determine the points of intersection of the tangent line and the curve forming the channel's boundary. These points of intersection mark the end of one range line and the beginning of the next. The activity served as a catalyst for discussion of the mean value theorem.

OBJECTIVES

The activity's mathematical objectives are to approximate curves by line segments, emphasize the derivative to calculate slope, calculate slope using the rise-over-run definition, and apply the point-slope form of a line. Students will also reinforce their geometry and algebra skills and will be introduced to ideas about concavity. Because the river channel is defined by a function and a vertical shift of the same function, this activity can be used to introduce or demonstrate the mean value theorem. A broader objective is to demonstrate the applicability of mathematics to real-world applications and introduce the idea that lines are often useful approximations of curves.

INTRODUCTION TO THE ACTIVITY

Range lights are used on bays, rivers, and other waterways to guide boats safely. They consist of two lighthouses that indicate a linear path—the range line—for the boat to follow. When the range lights are close to each other, the rear light is placed higher than the front light so that both are visible. The range lights also can be on opposite sides of the river, as on the Kennebec River in Maine. A boat follows one range line to its successor, where it will change direction, until it enters easily navigable waters, such as when a river ends in a larger body of deep water. On the activity sheets, the range lines are used to keep an imaginary boat between the channel curves $y = f(x)$ and $y = f(x) + h$. (See the range lines in **fig. 2**.) Although the parallel curves are an obvious simplification of a real-world situation, two such curves can always be constructed to represent a river's channel.

Guiding questions

For this activity to be completed successfully, students will use the definition of the slope of a line as well as the point-slope form of a line and will recall that the derivative evaluated at a point gives the slope of the tangent line of the associated function. When ready to begin, without mentioning the terms *secant* or *tangent*, hand out **sheets 1** and **3**. Do not hand out **sheet 2** until **sheet 1** is completed, because the language used in the questions on **sheet 2** reveals answers for some questions on **sheet 1**. It may be useful to introduce range lines by having students examine the Range Light Keepers Web site: www.rlk.org/.

EXPERIMENTING WITH SECANT LINES

To answer questions 1 and 2 on **sheet 1**, students draw different range lines to discover that the (first) optimal range line for river 1 begins at (0, 0) and is tangent to the channel's upper boundary curve. The point of tangency is found by equating the slopes obtained when doing the calculations in two different ways. The endpoint of the first range line becomes the starting point of the second range line, and the process of finding the optimal range line is iterated. Only two range lines are required for river 1.

For river 2, all the range lines except the second are found in a manner similar to that used for river 1. The second range line is not a secant line and therefore requires a different approach, as described below.

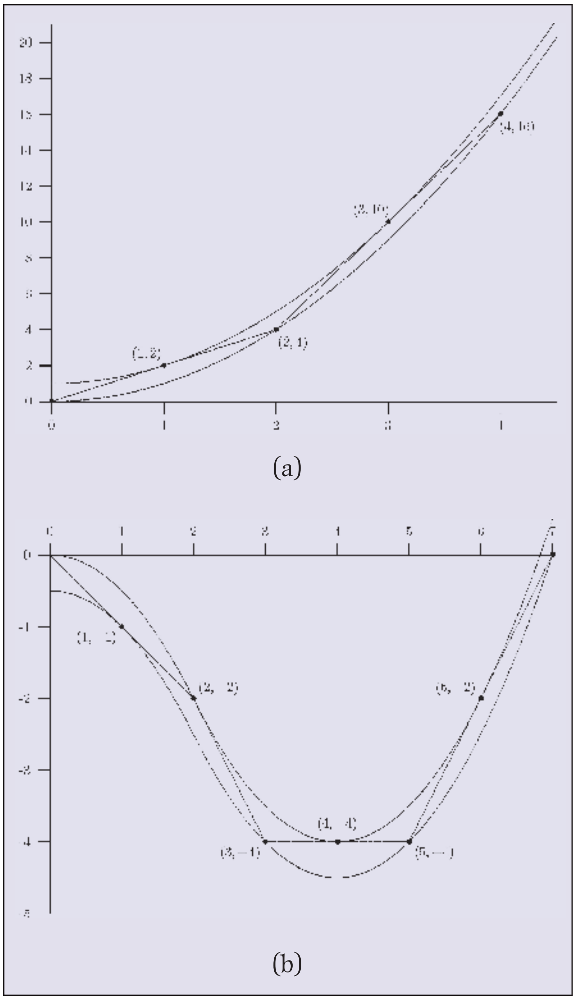


Fig. 2 Range lines, tangent points, and secant points for the channels of river 1 (**a**) and river 2 (**b**)

Guiding questions

Refrain from referring to the range line as a secant line or a tangent line until students have recognized that the lines may be secant to one curve and tangent to the other. Students develop and use vocabulary as part of the activity. It may be necessary to review how to calculate slope, because variables instead of values are used in the rise-over-run formula and the point-slope form of a line. Students can use river 1 to build intuition about concavity, which is discussed on **sheet 2**.

TANGENT LINES THAT ARE NOT SECANT LINES

After completing the first range line of river 2, students may feel confident in their abilities to construct range lines of maximum distance, because all range lines obtained so far were a secant line to one graph and a tangent line to the other graph. For the second range line of river 2, students encounter a situation that may cause some confusion. A change in concavity makes placing the range line more difficult. Students start the second range line on what appears to be the wrong side of the river: The channel is concave up, but the starting point of the second range line is on the upper curve. To construct the longest possible range line, the range line must be tangent to the upper curve at the starting point of the line. Consequently, the end of the second range line is on the lower curve. Because the channel is concave up and the starting point of the third range line is on the lower curve, the third range line will once again be a secant to one curve (lower curve) and tangent to the other (upper curve).

Guiding questions

Question 2 on **sheet 2** regarding river 2 is designed to get students to start thinking about concavity in order to help them construct the second range line. In question 3, students develop the necessary intuition about the optimal placement of the second range line. Question 4 reveals that the optimal range line is tangent to the upper curve. At this point, students must realize that tangency occurs at the starting point of the second range line. Students comfortable with concavity may need less guidance for the optimal placement of the second range line. **Sheet 2** could be used to introduce concavity.

EXTENSIONS

Quadratic polynomials were used as channel boundaries in rivers 1 and 2 to make the algebra simple by yielding integer solutions. The piecewise nature of river 2 introduced a change in concavity that was not present on river 1. Notice that the channel boundary curves are differentiable at $x = 2$. Any two curves $y = f(x)$ and $y = f(x) + h$ can form channel boundaries. And although other boundary curves may change concavity, students may need to use a calculator to determine the points of intersection and the endpoints of range lines.

For example, assume river 3 has its channel defined by third-degree polynomials $y = x^3 - 4x^2 + 4x$ and $y = x^3 - 4x^2 + 4x - 1$. The first range line begins on the upper curve at (0, 0), is tangent to the lower curve at (0.597, 0.175), and ends on the upper curve at (1.459, 0.427). As with river 2, river 3's concavity changes so that the second range line begins on the upper curve at (1.459, 0.427) and is tangent at that point to the upper curve. This range line ends on the lower curve at (2.347, –0.717). The third range line begins on the lower curve at (2.347, –0.717), intersects the upper curve at (2.844, 2.026), and ends on the lower curve at (3.293, 4.505).

For a concave-up or concave-down river, the slope of the range line is calculated by equating the slope of a tangent line of one curve and the slope of a secant line of the other curve. Because the two channel curves are vertical shifts of each other, relating the slope at the point of tangency to the slope of the secant line is an application of the mean value theorem because the derivative at any x is the same for the two curves. The mean value theorem applied to second-degree polynomials ensures that the tangent line evaluated at the midpoint between the two x-values that form the endpoints of a secant line has the same slope as the secant line.

For purposes of this activity, it is assumed that boats follow one range line as far as possible before picking up the next line. However, a boat might follow the next range line as soon as that range line intersects its current one. For river 2, it is possible that the second range line begins at the point of tangency of the first range line. Under these conditions, following the first and second range lines gets the boat farther along the river than under the solutions given below.

This activity reinforces concepts of slope and uses tangent lines and secant lines to construct a navigable path in a river channel. As such, the activity demonstrates how real-world problems relate to abstract mathematics so that students can understand mathematics on both a practical and a theoretical level. The activity also introduces a common theme in mathematics in which a curve is approximated by a straight line.

SOLUTIONS

Sheet 1: River 1

The endpoints of the range lines, as well as points of tangency, for river 1 appear in **figure 2a**.

1. For starting points (0, 1), (0, 1/2), and (0, 0), the longest range lines end at

$$(1,\ 1),\ \left(1+\frac{\sqrt{2}}{2},\ \frac{3}{2}+\sqrt{2}\right),\ \text{and}\ (2,\ 4),$$

   respectively. These are exact values and may not match a student's result obtained by using a ruler.

2. The maximum distance is achieved by starting at (0, 0). Using the minimum value of y in (0, y) as the starting point results in a range line of maximum slope. Because the boundary curves are concave up, a range line of maximum slope yields the maximum distance. Hence, the range line of maximum distance begins at (0, 0).

3. The longest range line is tangent to the curve $y = x^2 + 1$.

4. The longest range line is a secant of the curve $y = x^2$.

5. Using the points (0, 0) and (x^*, y^*), the slope is

$$\frac{(y^* - 0)}{(x^* - 0)} = \frac{\left[(x^*)^2 + 1\right]}{x^*}.$$

6. The range line is tangent to $y = x^2 + 1$ at (x^*, y^*). The slope of the tangent line is the derivative of $y = x^2 + 1$ evaluated at (x^*, y^*). Hence, the slope is $2x^*$.

7. Solving for x^* in

$$\frac{\left[(x^*)^2 + 1\right]}{x^*} = 2x^*$$

yields the point of tangency: $(x^*, y^*) = (1, 2)$.

8. The equation of the first range line is $y = 2x$. The range line intersects the channel's lower boundary $(y = x^2)$ at (2, 4).

9. The second range line has the equation $y = 6x - 8$. The second range line ends at (4, 16).

Sheet 2: River 2

The endpoints of the range lines, as well as points of tangency, for river 2 appear in **figure 2b**.

1. The longest possible first range line begins at (0, 0) and ends at (2, –2). This range line is tangent to the channel's lower boundary and is a secant line to the channel's upper boundary.

2. For concave-up boundary curves (as in river 1), the optimal starting point is on the lower curve. For concave-down boundary curves (as in river 2), the optimal starting point is on the upper curve. As an extension, consider asking students to check this rule of thumb on increasing curves that are concave down and decreasing curves that are concave up.

3. The second range line begins on the upper curve at a point on the river at which the boundary curves are concave up. According to the solution to question 2, the starting point is on the wrong side of the river.

4. The tangent line to the upper curve

$$y = \frac{(x-4)^2}{2} - 4$$

is $y = -2x + 2$. It intersects the lower curve at (3, –4).

5. The third range line begins at (3, –4) and ends at (5, –4); hence, it is tangent to the upper curve at (4, –4). The fourth range line begins at (5, –4) and ends at (7, 0); hence, it is tangent to the upper curve at (6, –2).

6. The range line approximates both boundary curves of a channel with the additional restriction that the segment must stay between the two curves. As an example, mapmakers often use a line segment to approximate a curved road. Further, straight-line distance is often used to approximate the distance along a road. As an example that appears later in the calculus, Newton's method approximates a zero of a nonlinear function by using the x-intercept of a straight line that approximates the function's graph.

BIBLIOGRAPHY

Boating Basics Online. Information about channel markers. www.boatingbasicsonline.com/course/boating/6_3.php.

Larson, R., R. P. Hostetler, and B. H. Edwards. *Calculus with Analytic Geometry.* 8th ed. Boston: Houghton Mifflin, 2005.

National Council of Teachers in Mathematics (NCTM). *Principles and Standards for School Mathematics.* Reston, VA: NCTM, 2000.

The Range Light Keepers. www.rlk.org/. ∞



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The Optimal Placement of Range Lights

Range lights consist of a pair of lighthouses on bays or rivers that form a line—the *range line*—for a boat to follow to ensure that it stays within the river channel. The boat continues along a range line until it either picks up the next range line or enters easily navigable waters.

River 1

Assume that the river channel is between the curves $y = x^2$ and $y = x^2 + 1$, as in **figure 1**. Answer the following questions about the sequence of longest possible range lines to help a boat stay in the channel. The longest range line is one that intersects the channel boundary farthest along the boat's course.

1. Use a ruler to draw the longest possible range lines for a boat entering the channel at $x = 0$ with the three different starting points: (0, 1), (0, 1/2), and (0, 0). Find the coordinates of the intersection.

2. Which starting point $(0, y)$ for y in the closed interval [0, 1] provides the maximum distance? Why? Explain.

3. The longest range line intersects $y = x^2 + 1$. Use calculus terminology to describe what is significant about this range line and this point of intersection.

4. Use calculus terminology to describe how this range line relates to the curve $y = x^2$.

5. Let (x^*, y^*) be the coordinates of the point of intersection of the range lines and $y = x^2 + 1$. Express the slope of this range line as a function of x^*.

6. Use the relationship (that you identified in question 3) between the range line and the curve $y = x^2 + 1$ to express the line's slope in an alternate way, also in terms of x^*.

7. Use the expressions for slope found in questions 5 and 6 to solve for coordinates of the point of intersection, (x^*, y^*).

8. Determine the equation of the range line. Find the second point at which the range line intersects $y = x^2$. This is the endpoint of the first range line.

9. Repeat this process using the endpoint of the first range line as the starting point of the second.
   (*a*) What is the equation of the second range line?
   (*b*) What is the endpoint of the second range line?

The Optimal Placement of Range Lights

Sheet 2

River 2

Consider a second river whose channel is defined by piecewise quadratic curves, as in **figure 2**. Answer the following questions about the sequence of longest possible range lines to help a boat stay in the channel.

$$y = \begin{cases} \dfrac{-x^2}{2} & \text{for } x \in [0, 2] \\ \dfrac{(x-4)^2}{2} - 4 & \text{for } x \in [2, 7] \end{cases} \quad \text{and} \quad y = \begin{cases} \dfrac{-x^2}{2} - \dfrac{1}{2} & \text{for } x \in [0, 2] \\ \dfrac{(x-4)^2}{2} - \dfrac{9}{2} & \text{for } x \in [2, 7]. \end{cases}$$

1. Starting with $x = 0$, determine the optimal placement of the first range line. How far can the boat travel safely in the x-direction? Discuss how this line relates to the two curves.

2. Compare the starting and ending positions for the first range lines of rivers 1 and 2. Explain how their optimal positioning depends on whether the curves are concave up or concave down.

3. Given your answer to question 1, use a ruler to draw the longest possible second range line. Does this range line fit either of the two cases discussed in question 2? You appear to begin on the wrong side of the river. Explain.

4. Use the tangent line of the upper curve as the next range line. At what point does this tangent line intersect the lower curve? Because the upper curve is concave up in this region, the tangent line lies below the curve.

5. Place the next two range lines in the optimal position and indicate the tangent and secant points.

6. Explain how the range line approximates a curve. Can you think of other instances in which a line approximates a curve?

From the May 2008 issue of MATHEMATICS teacher

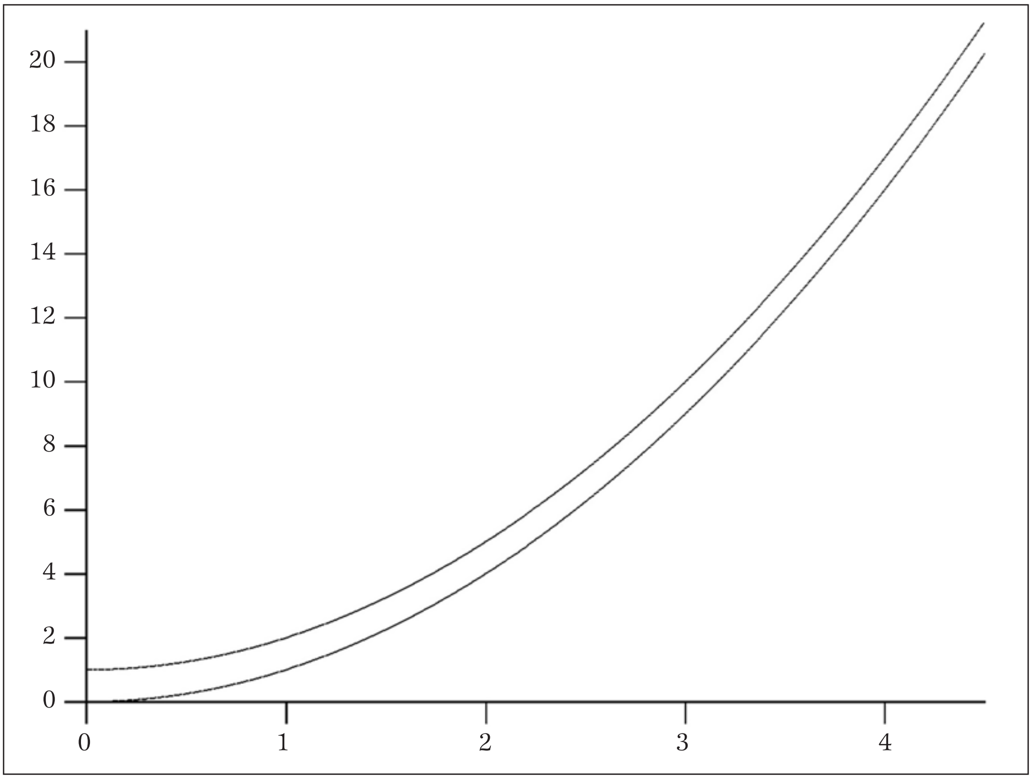

Fig. 1 River 1's channel

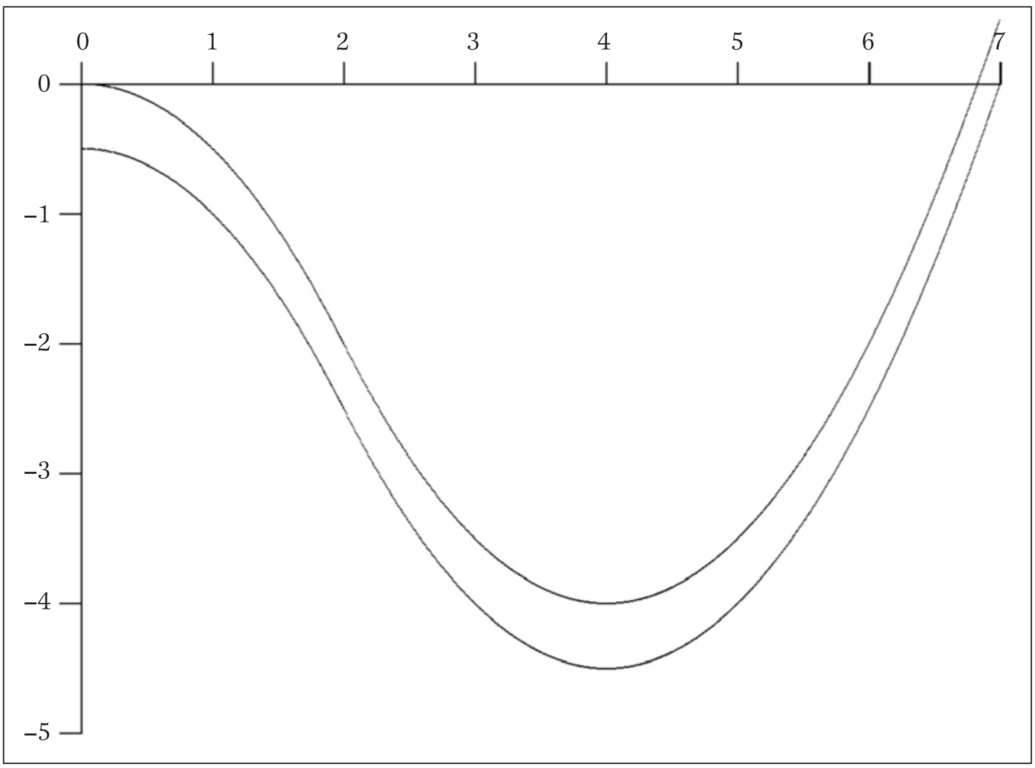

Fig. 2 River 2's channel